This exam has **two** problems:

- Problem 1 has four parts, and is worth 20 points.
- Problem 2 has five parts, and is worth 20 points. There is a bonus part in problem 2 which is worth 5 points.

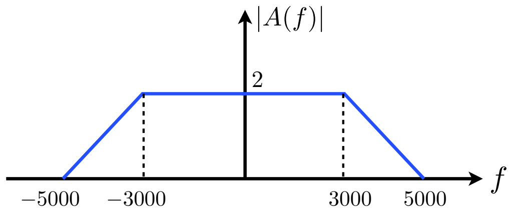
The Q-function is defined as

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

You can find the function evaluation for several values of x in the table below:

x	$Q(x)$
0	0.5000
0.1	0.4602
0.2	0.4207
0.3	0.3821
0.4	0.3446
0.5	0.3085
0.6	0.2743
0.7	0.2420
0.8	0.2119
0.9	0.1841
1.0	0.1587
1.1	0.1357
1.2	0.1151
1.3	0.0968
1.4	0.0808
1.5	0.0668
1.6	0.0548
1.7	0.0446
1.8	0.0359
1.9	0.0287
2.0	0.0228

Problem 1) Suppose a signal $A(t)$ is going to be transmitted over a channel. The magnitude of the Fourier transform of the $A(t)$ is given by



The maximum magnitude of $A(t)$ is 150: $|A(t)| \leq 150$. We use pulse coded modulation (PCM) to convert the continuous signal into a sequence of bits.

(a) What is the minimum sampling frequency required for perfect signal recovery? *[3 points]*

(b) We would like to design a uniform quantizer such that the average quantization noise power does not exceed 0.03. What is the maximum quantization interval width?

[8 points]

(c) What is the minimum number of quantization levels to satisfy the interval width requirement of part (b)? Assuming that the resulting sequence of bits is transmitted using BPSK, find the corresponding number of binary pulses per sample. *[5 points]*

(d) Let $x[n]$ be the binary sequence to be transmitted over the channel. To this end, first generate a baseband signal

$$x_b(t) = \sum_n x[n]\mathrm{sinc}(t/T - n),$$

and then modulate $x_b(t)$ using a carrier frequency f_c, that is multiplying it by $\cos(2\pi f_c t)$, and send it over a wireless channel. Determine the maximum value for T and the minimum value for f_c. *[4 points]*

Solution:

(a) From the power spectral density it is clear that $A(t)$ is a band-limited signal with bandwidth $W = 5$ kHz. For perfect recovery, we need to sample at least at Nyquist sampling rate, which is $f_s = 2W = 10$ kHz.

(b) Let a_{i-1} and a_i be two quantization boundaries, and $\hat{x}_i = \frac{a_{i-1}+a_i}{2}$ be the quantization point. We denote the interval length of each quantization bin by $\Delta = a_i - a_{i-1}$. For uniform distribution, we have

$$\text{Average Quantization noise} = \int_{a_{i-1}}^{a_i} \frac{1}{\Delta}(x-\hat{x}_i)^2 dx = \int_{-\frac{\Delta}{2}}^{+\frac{\Delta}{2}} \frac{1}{\Delta} u^2 du = \frac{\Delta^2}{12}.$$

In order to keep quantization noise below 0.03, we have

$$\frac{\Delta^2}{12} \leq 0.03 \quad \Rightarrow \quad \Delta \leq 0.6$$

(c) The total range of the signal is $2A_{\max} = 300$. If we use 2^ν quantization levels, we have

$$0.6 \geq \Delta = \frac{2A_{\max}}{2^\nu} = \frac{300}{2^\nu} \Rightarrow 2^\nu \geq 500 \Rightarrow \nu = \lceil \log_2 500 \rceil = 9.$$

That means we need $2^9 = 512$ quantization levels, and use 9 bits per sample.

(d) The total number of pulses need to be sent over a second is $10000 \times 9 = 90000$ pulses/second. Since the sinc functions in the summations are T seconds apart, we need to set T such that $90000T \leq 1$, or equivalently, $T \leq 1/90000\ s$.

In frequency domain, $X_b(f)$ will be limited to $[-1/2T, 1/2T]$, and therefore after multiplying by $\cos(2\pi f_c t)$, the resulting bandwidth would be $[-f_c - 1/2T, -f_c + 1/2T] \cup [f_c - 1/2T, f_c + 1/2T]$. In order to keep these intervals disjoint, we need $f_c > 1/2T \geq 45$ kHz.

Problem 2) Consider a Gaussian communication channel

$$Y(t) = X(t) + Z(t)$$

where $X(t) \in \{-1, +1\}$ and $Z(t)$ is an additive white Gaussian noise distributed as $Z(t) \sim \mathcal{N}(0, \sigma^2)$.

(a) What is the signal to noise ratio of this channel? *[2 points]*

Upon observing Y, the receiver detects the input signal using the following rule:

$$\hat{X} = \begin{cases} +1 & \text{if } Y \geq 0 \\ -1 & \text{if } Y < 0. \end{cases}$$

(b) Find the bit error rate of this channel, i.e., $\Pr(\hat{X} \neq X)$. *[4 points]*

(c) A source generates integer numbers in $\mathcal{S} = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Transmitter firsts maps the source to a binary sequence using the table below,

s	0	1	2	3	4	5	6	7
$b_1b_2b_3$	000	001	010	011	100	101	110	111

Table 1: The binary representation map.

and sends the sequence by transmitting one bit at a time over the channel, using $0 \mapsto -1$ and $1 \mapsto 1$. At receiver, we first decode the bits, and remap the binary sequence to $\hat{s} \in \mathcal{S}$. Find the probability that $\hat{s} = s$. *[3 points]*

(d) Find the average mean squared distortion of this communication channel, that is $\mathbb{E}[|s - \hat{s}|^2]$, when $s = 4$ is sent and $\sigma = 2/3$. *[6 points]*

(e) Assume we want to use a better map (compared to Table 1) to reduce the average mean squared error. What is a desired property of a good map? *[5 points]*

(f) Give a good mapping which satisfies the criteria suggested in part (e). *[Bonus: 5 points]*

Solution:

(a) It is clear that $\mathbb{E}[|X|^2] = \frac{1}{2}(-1)^2 + \frac{1}{2}(+1)^2 = 1$, and $\mathbb{E}[|Z|^2] = \sigma^2$. Therefore $SNR = \frac{1}{\sigma^2}$.

(b)

$$\begin{aligned}P_e &= \Pr(\hat{X} \neq X) = \Pr(\hat{X} = 1|X = -1)\Pr(X = -1) + \Pr(\hat{X} = -1|X = 1)\Pr(X = 1)\\ &= \Pr(Y \geq 0|X = -1)\Pr(X = -1) + \Pr(Y < 0|X = 1)\Pr(X = 1)\\ &= \Pr(Z \geq 1)\Pr(X = -1) + \Pr(Z < -1)\Pr(X = 1)\\ &= Q(1/\sigma)\Pr(X = -1) + Q(1/\sigma)\Pr(X = 1) = Q(1/\sigma).\end{aligned}$$

(c) s is sent over the channel in three channel uses. It can be perfectly decoded if no error occurs in any of the channel uses. Since transmission are performed independently, we have

$$\P(\hat{s} = s) = (1 - P_e)^3 = (1 - Q(1/\sigma))^3.$$

(d) Let $s = 4$ be the message to be transmitted. The actual binary sequence sent to the receiver would be 100. Depending on the occurring error in each channel use, the receiver decodes Therefore,

detected sequence	$\hat{s}$	$\lvert s-\hat{s}\rvert$	probability
000	0	4	$P_e(1-P_e)^2$
001	1	3	$P_e^2(1-P_e)$
010	2	2	$P_e^2(1-P_e)$
011	3	1	P_e^3
100	4	0	$(1-P_e)^3$
101	5	1	$P_e(1-P_e)^2$
110	6	2	$P_e(1-P_e)^2$
111	7	3	$P_e^2(1-P_e)$

$$\begin{aligned}\mathbb{E}[|s - \hat{s}|^2] &= \sum_{u\in\mathcal{S}} P(\hat{s} = u|s = 4)|4 - u|^2 = P_e^3 + 22P_e^2(1 - P_e) + 21P_e(1 - P_e)^2\\ &= 21P_e - 20P_e^2.\end{aligned}$$

When $\sigma = 2/3$, we have $P_e = Q(3/2) = Q(1.5) = 0.0668$. Therefore,

$$\mathbb{E}[|s - \hat{s}|^2] = 21P_e - 40P_e^2 = 1.3137$$

(e) In the binary expansion mapping given in part (c), we have $0 \mapsto 000$ while $4 \mapsto 100$. That means if the first bit gets corrupted by the noise, the detected symbol is very far from the transmitted one: $|\hat{s} - s|^2 = |0 - 4|^2 = 16$. A good mapping should assign codewords with small Hamming distance to symbols which are physically closed by, and vice versa.

(f) An example of such codes is the so called Gray code, given by

s	0	1	2	3	4	5	6	7
$b_1b_2b_3$	000	001	011	010	110	111	101	100